Towards Dynamic Probabilistic Logics: A Survey and a Proposal

Norihiro Ogata
Faculty of Language and Culture, Osaka University
1-8 Machikaneyama, Toyonaka, Osaka, Japan
ogata@lang.osaka-u.ac.jp

Abstract

In this paper, I will briefly survey probabilistic logics and investigate logics of probabilistic epistemic change, conjectures, and discovery of chances by resource-bounded agents in the sense of the limitation of the available data and its temporal instability, called dynamic probabilistic logic ($DPrL$), based on probabilistic dynamic logics and probabilistic Kripke systems, of which equivalence is defined by probabilistic bisimulation. $DPrL$ has actions of form of $[\varphi!]$ and probabilistic formula $(P \geq r)\varphi$. Furthermore, $DPrL$ will be extended by adding formulas of form of $[\exists F]\varphi$, a future operator over branching times, and formulas of form of $S_a\varphi$, of which meaning is 'happening of p is significant to agent a', defined by normalization of probability of p by agent a's expectation. The resulting system is called Dynamic Conjecture Logic (DCL), and in DCL we can describe 'subjective chance' in terms of the Chance Discovery.

Introduction

Logical treatment of agents' detection and management of fate and "chance"[1] involves nondeterminism and uncertainty of agents' epistemic dynamics, and therefore, it is natural that the logical treatment should be connected with probabilistic distributions of beliefs and probabilistic processes of revising them.

The treatment of probability within or in connection with logic have been one of the main tasks of logic. In mathematical logics or model theory, Keisler (Keisler 1985) proposes a logic of probability, in philosophical logics, since Carnap, many authors (cf. (Adams 1998)) propose many systems of probabilistic reasoning in computer science logics, modal logic (Desharnais, Edalat, & Panangaden 1998; Jonsson, Larsen, & Yi 2001), epistemic logic (Halpern & Tuttle 1993), temporal logics (Lehmann & Shelah 1982; Segala & Lynch 1994; Bianco & de Alfaro 1995), and dynamic logics (Feldman 1984; Feldman & Harel 1984; Kozen 1985) of probabilistic programs and probabilistic processes. Unfortunately, however, these probabilistic logics are not proposed for modeling the epistemic dynamics in terms of probability.

In this paper, I will briefly survey these probabilistic logics and investigate logics of probabilistic epistemic change, conjectures, and discovery of chances by resource-bounded agents in the sense of the limitation of the available data and its temporal instability, called *dynamic probabilistic logic* ($DPrL$).

In section 2, I will survey many logics related to probability and probability transitions, and then I will sum up how to express probabilistic information and probabilistic transitions in formal logics. In section 3, I will propose a semantics of $DPrL$, a dynamic logic (Goldblatt 1992) with belief change modality $[\varphi!]$ and probabilistic formula $(P \geq r)\varphi$, by combining a *Kripke system*, which is a reformulation of a Kripke model in terms of *coalgebras*, with a transition probabilistic function, called this a *probabilistic Kripke system*. In section 4, I will propose a dynamic logic of conjecture, prospect, and prediction by combining the basic $DPrL$ with future modality on branching times $[\exists F]$, formula 'happening of φ is significant to agent a', $S_a\varphi$, and probabilistic branching time models as in (Bianco & de Alfaro 1995), called *dynamic conjecture logic* (DCL). Lastly, in section 5, I will discuss the nature of "subjective chance" by considering some puzzles and tentatively propose a chance discovery in DCL.

Probability and Probabilistic Processes

Let $\mathcal{A}$ be a σ-algebra[2] of a set of individuals, numbers, or possible worlds X, and W be a set of possible worlds or states. The notions of *probabilistic measure*, *probability function*, *transition probability function*, and *Markov processes* are defined as follows:

1. $\mu : \mathcal{A} \to [0,1]$ is a probability distribution (or probability measure) on measurable space $\langle X, \mathcal{A}\rangle$, written

[1] I do not intend that this term mean (objective) probability, but rather I mean a kind of special situation that plays a role of a key of agents' decision of thier future actions. In section 5, I will define this term more explicitly.

[2] $\mathcal{A}$ is a σ-algebra of X if (i) $\emptyset \in \mathcal{A}$, (ii) $X_i \in \mathcal{A} \Rightarrow \bar{X}_i (= X - X_i) \in \mathcal{A}$ (complement), (iii) $X_i \in \mathcal{A}$ $(n \geq 1) \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ (countable union) (Halmos 1950).

$\mu \in \mathcal{D}(X)$ or $\mathcal{D}(X, \mathcal{A})$, if it is a mesurable space, i.e., pair $\langle X, \mathcal{A}\rangle$, satisfying the following conditions:

(a) it is countably additive, i.e., if for every disjoint sequence $\{X_i\}_{i\in I}$ of sets such that $\bigcup_{i=0}^{\infty} X_i \in E$, then $\mu(\bigcup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} \mu(X_i)$,

(b) it is positive, i.e., $\mu(Y) \geq 0$ for all $Y \in \mathcal{A}$, and

(c) $\mu(X) = 1$.

2. μ is a probability function if it satisfies the following conditions, called Kolmogorov's Axioms:

K1 $0 \leq \pi(A) \leq 1$ ($\approx$ (b))

K2 $\pi(W) = 1$ (=(c))

K3 $A \subseteq B \Rightarrow \pi(A) \leq p(B)$

K4 $A \cap B = \emptyset \Rightarrow \pi(A \cup B) = \pi(A) + \pi(B)$ (finite additivity[3]$\approx$(a))

3. $\rho : X \times X \to [0, 1]$ is a transition probability function over X, if for each $w \in X$, $\sum_{u\in X} \rho(w, u) = 1$.

4. $\rho : X \times \mathcal{A} \to [0, 1]$ is also called a transition probability function over X if for each $x \in X$, the function $\rho(x, \cdot)$ is a subprobability measure, and for each $Y \in \mathcal{A}$, $\rho(\cdot, Y)$ is a measurable function.

5. $\langle X, \rho\rangle$ is a Markov process.

A Survey of Probabilistic Logics

Probabilistic Quantifier Logic

Keisler (Keisler 1985) introduces a logic with probability quantifier $\mathcal{L}_{\mathbb{A}P}$. $\mathcal{L}_{\mathbb{A}}$ is the infinitary logic, where $\mathbb{A}$ is an admissible set (possibly with urelements) such that $\omega \in \mathbb{A}$ and each $a \in \mathbb{A}$ is countable, and $L_{\mathbb{A}P}$ possesses the probability quantifier $(Px \geq r)$. That is, a new formula $(Px \geq r)\varphi(x)$ is added and it means that the set $\{x|\varphi(x)\}$ has probability at least r. A formula $\varphi \in L_{\mathbb{A}P}$ is defined as follows:

$$\varphi ::= R(\tau_1, \ldots, \tau_n)|\neg\varphi| \bigwedge \Gamma|(P\bar{x} \geq r)\varphi$$

where $\Gamma \in \mathbb{A}$, $L_{\mathbb{A}P} \subseteq \mathbb{A}$, $r \in \mathbb{A} \cap [0, 1]$, $\bar{x} = \langle x_1, \ldots, x_n\rangle$, and $L_{\mathbb{A}P} = \mathbb{A} \cap L_{\omega_1 P}$.

A probability structure for $L_{\mathbb{A}P}$ is a structure

$$\langle D, R_i^{\mathcal{M}}, c_j^{\mathcal{M}}, \mu\rangle_{i\in I, j\in J},$$

where μ is a (countably additive probability measure on D such that each singleton is measurable[4], each $R_i^{\mathcal{M}}$ is $\mu^{(n_i)}$-measurable[5], and each $c_j^{\mathcal{M}} \in D$.

$$\mathcal{M}, g \models (P\bar{x} \geq r)\varphi(\bar{x}) \text{ iff } \mu^{(n)}([\![\varphi]\!]^{D,g,\bar{x}}) \geq r$$

[3] A (extended) real valued function μ is *finitely additive* if, for every finite, disjoint class $\{X_i\}_{i<n}$ of sets such that $\bigcup_{i<n} X_i \in E$, then $\mu(\bigcup_{i<n} X_i) = \sum_{i<n} \mu(X_i)$ (Halmos 1950).

[4] Given a σ-algebera $\mathcal{A}$, a set $Y \in \mathcal{A}$ is called a *measurable set*, *event*, or *μ-measurable*.

[5] A probability space $\langle X^n, \mathcal{A}^{(n)}, \mu^{(n)}\rangle$ ($n \in \omega$) is the *n-fold product probability space* such that $\mu^{(n)}(\times_{i<n+1} X_i) = \prod_{i<n+1} \mu(X_i)$, where $X_i \subseteq X_{i+1}$ and $\mu(X_{i+1}) = 0$ for any $i < n + 1$. $\mathcal{A}^{(n)}$ is called the σ-algebra generated by the set of measurable rectangles $\times_{i<n+1} X_i$ (Keisler 1985).

$(P\bar{x} < r)\varphi \equiv \neg(P\bar{x} \geq r)\varphi$, $(P\bar{x} \leq r)\varphi \equiv (P\bar{x} \geq 1 - r)\neg\varphi$, and $(P\bar{x} > r)\varphi \equiv \neg(P\bar{x} \geq 1 - r)\neg\varphi$.

$\mathcal{L}_{\mathbb{A}P}$ is the smallest set of $L_{\mathbb{A}P}$-formulas satisfying the following axioms and inference rules:

A1 All axioms of $\mathcal{L}_{\mathbb{A}}$ without quantifiers.

A2 $\bigwedge_{s\in[0,1]} \bigwedge_{r\in[0,1]} r \geq s \to ((P\bar{x} \geq r)\varphi \to (P\bar{x} \geq s)\varphi)$ (monotonicity (cf. Kolmogorov's K3)

A3 $(P\bar{x} \geq r)\varphi(\bar{x}) \to (P\bar{y} \geq r)\varphi(\bar{y})$

A4 $(P\bar{x} \geq 0)\varphi$

A5 finite additivity

(i) $(P\bar{x} \leq r)\varphi \wedge (P\bar{x} \leq s)\psi \to (P\bar{x} \leq r + s)(\varphi \vee \psi)$

(ii) $(P\bar{x} \geq r)\varphi \wedge (P\bar{x} \geq s)\psi \wedge (P\bar{x} \leq 0)(\varphi \wedge \psi) \to (P\bar{x} \geq r + s)(\varphi \vee \psi)$

A6 $(P\bar{x} > r)\varphi \equiv \bigvee_{n\in\mathbb{N}}(P\bar{x} \geq r + \frac{1}{n})\varphi$ (the Archimedean property)

B1 $\bigwedge_{\Psi\in Fin(\Theta)}(P\bar{x} \geq r) \bigwedge \Psi \to (P\bar{x} \geq r) \bigwedge \Theta$ (countable additivity)

B2 $(P\bar{x} \geq r)\varphi \to (P\theta\bar{x} \geq r)\varphi$, where θ is a permutation of $\{x_i\}_{i<n}$ (symmetry)

B3 $(P\bar{x} \geq r)(P\bar{y} \geq s)\varphi \to (P\bar{x}\bar{y} \geq r \cdot s)\varphi$, where $\bar{x}, \bar{y}$ are distinct. (product independence)

B4 $\bigwedge_{r<1}(P\bar{x} \geq 1)(P\bar{y} \geq 0)(P\bar{z} \geq r)(\varphi(\bar{x}\bar{z}) \equiv \varphi(\bar{y}\bar{z}))$, where $\bar{x}, \bar{y}, \bar{z}$ are distinct. (product measurability)

R1 Modus Ponens

R2 Conjunction: $\{\varphi \to \psi|\psi \in \Psi\} \vdash \varphi \to \bigwedge \Psi$

R3 Generation: $\varphi \to \psi(\bar{x}) \vdash \varphi \to (P\bar{x} \geq 1)\psi(\bar{x})$

The minimal logic satisfying A1-6+R1-3 is called *weak* $\mathcal{L}_{\mathbb{A}P}$ and the minimal logic satisfying A1-6+B1-4+R1-3 called (full) $\mathcal{L}_{\mathbb{A}P}$.

Propositional Probabilistic Dynamic Logics

I introduce two propositional probabilistic dynamic logics: PPrDL and PrPDL.

PPrDL is a propositional probabilistic dynamic logic proposed by Feldman (Feldman 1984), which has frequency terms, and assigns measure revision functions to programs as their meanings. For example, $Fr(\phi)$ denotes the frequency of the event denoted by the formula ϕ. $Pr(\phi)$, i.e., $\frac{Fr(\phi)}{Fr(\top)}$, a probability term that denotes the probability of the event denoted by ϕ, is derived from frequency terms as the following definition:

$$\begin{aligned} Pr(\phi) = v \quad \equiv \quad & (Fr(\top) \neq 0 \to v \cdot Fr(\phi)) \\ & \wedge(Fr(\top) = 0 \to v = 0) \end{aligned}$$

That is, frequency can be considered as unnormalized probability.

Instead of the notations $Fr(\phi) = v$ and $v < Fr(\phi)$, I write them as $(Fr = v)\phi$ and $(Fr > n)\phi$, respectively, since we compare PPrDL with $\mathcal{L}_{\mathbb{A}P}$.

Let $v \in GVar$ a global variable, $\alpha \in \Pi$ a set of atomic programs, and $p \in \Phi_0$ a set of atomic event formulas. Then, the language of PPrDL is defined as follows:

$\pi ::= \alpha|\pi_1;\pi_2|\pi_1+\pi_2|\pi^\infty|\phi?$
(programs $\pi \in Prog(\Phi,\Pi)$)

$\phi ::= p|\phi_1 \vee \phi_2|\neg\phi$
(event formulas $\phi \in \Phi$)

$\varphi ::= (\tau_1 < \tau_2)|(\tau_1 = \tau_2)|\varphi_1 \vee \varphi_2|\neg\varphi|\exists v\varphi|[\alpha]\varphi|(Fr = r)\phi|(Fr > r)\phi$
(formulas $\varphi \in L(\Phi,\Pi)$)

Let $D \subseteq \mathbb{R}$, $\mathcal{A}$ a σ-algebra of subsets of D, and M be the set of measures on measurable space $\langle D, \mathcal{A}\rangle$. A model of PPrDL is a structure

$$\mathcal{M} = \langle D, \mathcal{A}, \pi, \{[\![\alpha]\!]\}_{\alpha \in Prog(\Phi,\Pi)}, g\rangle,$$

where $\pi : \mathcal{A} \to \mathbb{R}$, an unnormalized probabilistic function, $[\![\alpha]\!] : M \to M$, an revision function of measures, and $g : GVar \to \mathbb{R}$, satisfying the following conditions:

- For all $\mu \in M$, $[\![\alpha]\!]^M(\mu)(D) \leq \mu(D)$.
- $[\![\phi?]\!](\mu) = \mu_{[\![\phi]\!]}$,
  where $\mu_A(B) = \begin{cases} 0 & \text{if } A \cap B = \emptyset \\ \mu(A \cap B) & \text{otherswise} \end{cases}$.
- $[\![\pi_1;\pi_2]\!] = [\![\pi_2]\!] \circ [\![\alpha_1]\!]$
- $[\![\pi_1 + \pi_2]\!] = [\![\pi_1]\!] + [\![\pi_2]\!]$
- $[\![\pi^\infty]\!] = \sum_{k=0}^{\infty}([\![\pi]\!])^k$
- For each term τ, $[\![\tau]\!] : M \to \mathbb{R}$
- For each $p \in \Phi_0$, $[\![p]\!] \in \mathcal{A}$
- $[\![\neg\phi]\!] = D - [\![\phi]\!]$
- $[\![\phi_1 \vee \phi]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$

The truth condition of the formulas is defined as follows:

$$\mathcal{M}, \mu \models [\pi]\varphi \Leftrightarrow \mathcal{M}, [\![\alpha]\!]^{\mathcal{M}}(\mu) \models \varphi$$
$$\mathcal{M}, \mu \models \tau_1 = \tau_2\phi \Leftrightarrow [\![\tau_1]\!](\mu) = [\![\tau_2]\!](\mu)$$
$$\mathcal{M}, \mu \models \tau_1 < \tau_2\phi \Leftrightarrow [\![\tau_1]\!](\mu) < [\![\tau_2]\!](\mu)$$
$$\mathcal{M}, \mu \models (Fr > r)\phi \Leftrightarrow \mu(\pi([\![\phi]\!])) > r$$
$$\mathcal{M}, \mu \models (Fr = r)\phi \Leftrightarrow \mu(\pi([\![\phi]\!])) = r$$
$$\mathcal{M}, \mu \models \exists v\varphi \Leftrightarrow \exists r \in \mathbb{R}.\mathcal{M}[r/v], \mu \models \varphi$$

The logic of PPrDL is the smallest set of PPrDL-formulas satisfyin the following axioms and inference rules:

A0 All axiom schemes of the first-order predicate calculus

A1 $([\pi]\varphi) \equiv \varphi$ for φ containing no occurrence of Fr.

A2 $[\pi](\varphi_1 \vee \varphi_2) \equiv [\pi]\varphi_1 \vee [\pi]\varphi_2$

A3 $[\pi]\neg\varphi \equiv \neg[\pi]\varphi$

A4 $[\pi]\exists v\varphi \equiv \exists v[\pi]\varphi$

A5 $[\pi_1;\pi_2]\varphi \equiv [\pi_1][\pi_2]\varphi$

A6 $[(\phi?;\pi_1) + (\neg\phi?;\pi_2)](Fr = v)\varphi \equiv \exists u([\phi??;\pi_1](Fr = u)\varphi \wedge [\neg\phi??;\pi_2](Fr = v-u)\varphi$

A7 $[\phi_1??](Fr = v)\phi_2 \equiv (Fr = v)(\phi_1 \wedge \phi_2)$

A8 $([(\phi?;\pi)^\infty;\neg\phi?]\varphi) \equiv ([(\phi?;(\pi;((\phi?;\pi)^\infty;\neg\phi?)) + (\neg\phi?;\top?))]\varphi)$

A9 $([\top?]\varphi) \equiv \varphi)$

R1 Modus Ponens

R2 Generalization on $[\alpha]$ and $\forall$

where $(\phi??) = ((\neg\phi?;\top?)^\infty;\phi?)$, which I call a *strong test*.

Although this axiomatization's completeness is still open, PPrDL is decidable (Feldman 1984). PPrDL is a propositional fragment of PrDL (Feldman & Harel 1984), which corresponds to First-order Dynamic Logic (DL) (Goldblatt 1992), and handles random assignments of variables. PrDL is sound and complete (Feldman & Harel 1984).

However, whereas the formal properties of PPrDL and PrDL are very fine, the meanings of programs in them are revision functions of probabilistic distribution. As we will see the next subsections, it is natural that they are regarded as probabilistic transitions. Kozen (Kozen 1985)'s PPDL is such a probabilistic dynamic logic.

In PPDL, a proposition p generalizes to a measurable function[6] f. A state s generalizes to a measure μ. $s \models p$ generalizes to an integral $\int p d\mu$[7], a real-valued function giving the probability that state μ satisfies proposition φ.

The language of PPDL is defined as follows:

$\pi ::= \alpha|\pi_1;\pi_2|r_1\pi_1 + r_2\pi_2|\pi^*|\phi?$
(programs $\pi \in Prog(\Phi,\Pi)$)

$\phi ::= 1|p|r_1p_1 + r_2p_2|\phi \cdot p|\langle\pi\rangle\phi$
(function term $\phi \in F(\Phi,\Pi)$)

$\varphi ::= p_1 \leq p_2$ (formula $\varphi \in L(\Phi.\Pi)$)

$$\varphi_1 \vee \varphi_2 = \varphi_1 + \varphi_2 - \varphi_1 \cdot \varphi_2$$
$$\neg p = 1 - p$$
$$0 = \neg 1$$

A model

$$\mathcal{M} = \langle W, \mathcal{A}, [\![,]\!]\rangle$$

where W is a set of states, $\mathcal{A}$ a σ-algebra over W, and

$$[\![\pi]\!] : W \times \mathcal{A} \to \mathbb{R}$$

[6] Let $\langle X, \mathcal{A}\rangle$ be a measurable space. For every (extended) real valued function f on X and for every Borel subset M of the real line the set $\{x|f(x) \neq 0\} \cap f^{-1}(M)$ is measurable, then f is called a *measurable function* (Halmos 1950).

[7] That is, if f is simple and $f = \sum_{i=1}^{n}\alpha_i\chi_{E_i}$ on a measure space $\langle X, \mathcal{A}, \mu\rangle$,

$$\int f d\mu = \sum_{i=1}^{n}\alpha_i\mu(E_i).$$

f is simple if there is a finite, disjoint class $\{E_i\}_{i<n}$ of measurable sets and a finite set $\{\alpha_i\}_{i<n}$ of real numbers such that

$$f(x) = \begin{cases} \alpha_i & \text{if } x \in E_i \\ 0 & \text{if } x \notin \bigcup_{i<n} E_i \end{cases}$$

The simplest case of a simple function is the characteristic function χ_E of a measurable set E, i.e., $f(x) = \sum_{i=1}^{n}\alpha_i\chi_{E_i}(x)$ (Halmos 1950).

a positive, total Markov transition on $W \times \mathcal{A}$, i.e., for any $s \in W$ and $A \in \mathcal{A}$, $s\,[\![\pi]\!]A \geq 0$ and $s\,[\![\pi]\!]W = 1$.

$$[\![(r_1\pi_1 + r_2\pi_2)]\!] = [\![r_1\pi_1]\!] + [\![r_2\pi_2]\!]$$
$$[\![(\pi_1;\pi_2)]\!] = \lambda s\lambda C. \int_{t\in W} [\![\pi_2]\!](t, C)\,[\![\pi]\!](s, dt)$$
$$[\![\phi?]\!] = \lambda s\lambda C.C(s) \cdot [\![\phi]\!](s)$$
$$[\![(r_1p_1 + r_2p_2)]\!] = [\![r_1p_1]\!] + [\![r_2p_2]\!]$$
$$[\![\phi p]\!] = [\![\phi]\!] \cdot [\![p]\!]$$
$$[\![\langle\pi\rangle p]\!] = [\![\pi]\!] \cdot [\![p]\!]$$

$PPDL$ has the finite model property and a polynominal-space algorithm of decision procedure (Kozen 1985).

Probabilistic Computational Tree Logics

Probabilistic computational tree logics are probabilistic extension of Computational Tree Logic (CTL), i.e., a temporal logic of branching times with special formulas of form of

$$\forall(\varphi_1 \mathcal{U} \varphi_2).$$

A model is a tuple:

$$\mathcal{M} = \langle W, \longrightarrow, V\rangle,$$

where $\longrightarrow$ is serial, i.e., $\forall s \exists t.s \longrightarrow t$, read as "$t$ is a possible immediate successor to s". An $\longrightarrow$-branch σ starting at s in $\mathcal{M}$, $\sigma \in Branch(s)$ is an infinite sequence $s_0 \ldots s_n \ldots$ with $s = s_0$ and $s_n \longrightarrow s_{n+1}$ for all n. The semantics of the formula is defined as follows:

$$\mathcal{M}, s \models \forall(\varphi_1 \mathcal{U} \varphi_2) \;\Leftrightarrow\; \forall\sigma \in Branch(s).\exists k.\mathcal{M}, s_k \models \varphi_2 \wedge \forall i.0 \leq i < k \Rightarrow \mathcal{M}, s_i \models \varphi_1$$

Some of the other formulas and their informal meaning are defined as follows:

- $[\forall F]\varphi \equiv \forall(\top \mathcal{U} \varphi)$ ... for all path σ, there is a time σ_i such that φ holds at σ_i
- $[\forall G]\varphi \equiv \forall\neg(\top \mathcal{U} \neg\varphi)$ ... for all path σ, for every time σ_i, φ holds at σ_i
- $[\exists F]\varphi \equiv \neg\forall\neg(\top \mathcal{U} \varphi)$... for some path σ, there is a time σ_i such that φ holds at σ_i
- $[\exists G]\varphi \equiv \neg\forall(\top \mathcal{U} \neg\varphi)$... for some path σ, for every time σ_i, φ holds at σ_i

Probabilistic CTLs contain probabilistic formula of form of

$$(Pr \geq r)\varphi$$

adding to CTL.

PCTL (Segala & Lynch 1994) is a probabilistic CTL based on labelled Markov chains. Let $\langle W, \rho, L\rangle$ be a labelled Markov chain. A path is a (in)finite sequence $\sigma = s_0 s_1 \ldots$ where $s_i \in W$ and $\rho(s_i, s_{i+1}) > 0$ for $i = 0, 1, 2, \ldots$. $Path_\omega(s) = \{\sigma | \sigma = s_0 s_1 \ldots, s_0 = s\}$. $\mathcal{A}_s$ is the smallest σ-algebra on $Path_\omega(s)$ containing the basic cylinders $\{\sigma \in Path_\omega(s) | \sigma = \tau^\wedge\sigma'\}$ for $|\tau| < \omega$. The probability measure μ on $\langle Path_\omega(s), \mathcal{A}_s\rangle$ is the unique measure with

$$\mu(\{\sigma \in Path_\omega(s) | \sigma = \tau^\wedge\sigma'\}) = \prod_{0 \leq i < |\tau|} \rho(\tau_i, \tau_{i+1}).$$

The truth condition of the probabilistic formulas is defined as follows:

$$s \models (Pr \geq r)\varphi \Leftrightarrow \mu(\{\sigma \in Path_\omega(s) | \sigma \models \varphi\}) \geq r.$$

pCTL is a probabilistic version of CTL, based on *probabilistic-nondeterministic systems* (PNS), proposed by (Bianco & de Alfaro 1995). Given a set of states W and a set of atomic propositions Φ, a PNS over W and Φ is a quadruple

$$\mathcal{M} = \langle W, \longrightarrow, s_0, V\rangle,$$

where $s_0 \in W$ is the initial state, $V : W \to pow(\Phi)$ a valuation, $\longrightarrow$ a function that associates with each $s \in W$ the set $s \longrightarrow \{\pi_i\}_{i<|s\longrightarrow|}$ of the probability distributions over W next from s. I write $s \longrightarrow \pi$ if $\pi \in s \longrightarrow$. A *reachability* relation $\rightsquigarrow \subseteq W \times W$ is defined by

$$s \rightsquigarrow t \Leftrightarrow \exists\pi.s \longrightarrow \pi^s \wedge \pi(t) > 0.$$

The set of *legal infinite sequences* of S beginning at s is defined by

$$\Omega_s = \{s_0 s_1 s_2 \ldots | s = s_0 \wedge \forall n \in \mathbb{N}.s_n \rightsquigarrow s_{n+1}\}$$

The semantics of the probabilistic formula in pCTL is defined as follows:

$$s \models (Pr \geq r\varphi) \Leftrightarrow \mu_s^-(\{w \in \Omega_s | \omega \models \varphi\}) \geq r,$$

where $\mu_s^-(X) = \inf_\eta \mu_{s,\eta}(X)$, where η is a *strategy* (see (Bianco & de Alfaro 1995)).

Probabilistic Modal Logics (PML) I introduce two PMLs, modal logics with formulas of form of $\langle\alpha\rangle_r\varphi$ (or $\langle\alpha\rangle\langle r\rangle\varphi$). PMLs are used for logical characterization of bisimulation of probabilistic transition processes.

One of them is based on *probabilistic transition systems* (in the sense defined below), proposed by (Jonsson, Larsen, & Yi 2001).

$$\mathcal{M} = \langle W, \longrightarrow, \pi_0\rangle$$

is a probabilistic transition system if $\longrightarrow \subseteq W \times Act \times \mathcal{P}(W)$ and $\pi_0 \in \mathcal{P}(W)$ is the initial probability distribution. w is an initial state of $\mathcal{M}$ if $\pi(w) > 0$. w is reachable in $\mathcal{M}$ if there exists a sequence $(w_0, \ldots, w_n)$ such that w_0 is an initial state, $w_n = w$, and $\forall_{0\leq i<n} \exists\pi_{i+1}.w_i \xrightarrow{a_i} \pi_{i+1} \wedge \pi_{i+1}(w_{i+1}) > 0$. $w \overset{a}{\rightsquigarrow} w'$ iff $\exists\pi.s \xrightarrow{a} \pi \wedge \pi(w') > 0$. $\mathcal{M}$ is called finite if $|W| < \omega$ and $\rightsquigarrow$ is acyclic. If $w \xrightarrow{a} \pi_1$ and $w \xrightarrow{a} \pi_2$ implies $\pi_1 = \pi_2$, then $\mathcal{M}$ corresponds to a Markov decision process.

$$\mathcal{M}, w \models \langle a\rangle_r\varphi \Leftrightarrow \exists\pi.w \xrightarrow{a} \pi \wedge \pi([\![\varphi]\!]^W) \geq r$$

Another of them is based on *labeled Markov processes*, proposed by (Desharnais, Edalat, & Panangaden 1998).

A *partial labeled Markov process* with label set *Act* is a structure

$$\mathcal{M} = \langle W, \mathcal{A}, \{[\![\alpha]\!]\}_{\alpha\in Act}\rangle,$$

where W is the set of states, which is assumed to be an *analytic space*, and $\langle W, \mathcal{A}\rangle$ is the Borel σ-field[8] on W, and $[\![\alpha]\!] : W \times \mathcal{A} \to [0,1]$ is a transition sub-probability function[9].

The semantics of the formula is defined as follows:

$$\mathcal{M}, w \models \langle\alpha\rangle_r\varphi \quad\Leftrightarrow\quad \exists X \in \mathcal{A}.\forall u \in X.\mathcal{M}, u \models \varphi \\ (w\,[\![\alpha]\!]X) > r$$

Since, in DPrL, as DEL, the epistemic states are characterized by *bisimulation*, the results of PMLs are quite useful for designing the language and its semantics of DPrL.

A Proposal: Probabilistic Kripke Systems and Dynamic Probabilistic Logics ($DPrL$)

$DPrL$-Formulas

In this section I propose DPrL based on *probabilistic Kripke systems*. $DPrL$ has a generalized update modality $[\varphi!]$, a probabilistic nondeterministic choice $[\alpha_{1r} + \alpha_2]$, and a probabilistic modality $\langle r\rangle$, adding to PDL. For example, with these expressions, $DPrL$ can express the following expressions.

1. $[\varphi!](P \geq 0.8)\psi$ (by data φ, the chance of event ψ is more than 80%)
2. $(P \geq 0.8)[\varphi!]\psi$ (the chance of event ψ when φ holds is more than 80%)
3. $(P \geq 0.8)[\alpha_1; \ldots; \alpha_n]\top$ (the chance of sequence $\alpha_1 \ldots \alpha_n$'s being acceptable is more than 80%)

Formula 1 expresses that getting data φ is a chance of achieving ψ. Formula 2 expresses that getting data φ must be a chance of achieving ψ. Formula 3 can also be regarded as expressing a Hidden Markov Model (HMM)(Rabiner & Juang 1986; Rabiner 1989; Charniak 1993).

Probabilistic Kripke Systems (PKS)

A PKS is an extension of a Kripke System, which is a reformulation of a Kripke model by using *coalgebras*, defined as follows.

Definition 1 *Let W be a set, Φ a set of atomic propositions, Act a set of actions, and Γ a monotone functor.*

1. *$\langle W, e\rangle$ is a Γ-coalgebra if $e : W \to \Gamma(W)$.*
2. *$\langle W, e\rangle$ is a Γ-Kripke system if it is a Γ-coalgebra, where $\Gamma : X \mapsto pow(\Phi) \times pow(Act \times X)$.*
3. *$\mathcal{P} = \langle W, e, \{\rho_\alpha\}_{\alpha\in Act}\rangle$ is a probabilistic Kripke system if $\langle W, e\rangle$ is a Γ-Kripke system on W and $\rho(\alpha) : W \times W \to [0,1]$ is a Markov process over W, satisfying the following conditions:*
   - $\rho(\alpha, w, u) \in (0,1]$ *iff* $\langle\alpha, u\rangle \in \pi_2(e(w))$, *for any* $\alpha \in Act$,
   - $\rho(\varphi?)(w,u) = 1$ *iff* $w = u$ *and* $\mathcal{P}, w \models \varphi$,
   - $\rho(p?)(w,w)$ *is defined for all* $p \in \Phi$,
   - $\rho(\alpha)(w,u)$ *is defined for all* $\alpha \in \Pi$,
   - $\rho(\varphi_1 \vee \varphi_2?)(w,w) = \rho(\varphi_1)(w,w) + \rho(\varphi_2)(w,w)$ *if* $\{w|\rho(\varphi_1?)(w,w), \rho(\varphi_2)(w,w)\} \neq \emptyset$,
   - $\rho([\alpha]\varphi?)(w,w) = \sum\{\rho(\alpha,w,u)|\rho(\alpha,w,u) > 0 \to \rho(\varphi)(u,u) = 1\}$,
   - $\rho((P \geq r)\varphi?)(w,w) = \begin{cases} 1 & \textit{if } (\varphi?, w, w) \geq r \\ 0 & \textit{otherwise} \end{cases}$.

   The class of Γ-probabilistic Kripke models is denoted by $PKS(\Gamma)$.

As well as Kripke Systems, an equivalence among PKSs is characterized by bisimulation. Since Larsen & Skou (Larsen & Skou 1991), some authors have proposed the notion of *probabilistic bisimulations*.[10] The probabilistic bisimulation among PKSs is defined as follows.

Definition 2 *Let $\mathcal{P} = \langle W, e, \rho\rangle$, $\mathcal{Q} = \langle U, f, \sigma\rangle \in PKS(\Gamma)$, $w \in W$, and $u \in U$.*

[8] $\langle X, \mathcal{A}\rangle$ is a Borel σ-field if it is a topological space and measurable space.

[9] A *subprobability measure* μ if it is positive and $\mu(X) \leq 1$ (Halmos 1950).

[10] de Vink and Rutten (de Vink & Rutten 1997) proposes discrete probabilistic transition systems and thier probabilistic bisimulation, as follows:
A discrete probabilistic transition system is a tuple $\langle Pr, Act, \rho\rangle$ where Pr is a given set of processes, Act a given set of actions, and $\rho : Pr \times Act \times Pr \to [0,1]$ is a transition probability function. A probabilistic bisimulation for a discrete probabilistic transition system is an equivalence $\simeq$ on Pr such that

$$\forall P, Q \in Pr.a \in Act.E \in P/\simeq.P \simeq Q \quad\Rightarrow \\ \sum_{P'\in E}\rho(P,a,P') = \sum_{P'\in E}\rho(Q,a,P')$$

Stark & Smolka (Stark & Smolka 2000) proposes finite-state probabilistic processes and their probabilistic bisimulation, as follows:
A probabilistic process $\alpha \in Pr$ is defined as follows:

$$\alpha ::= x|a\alpha|\alpha_{1r} + \alpha_2|\mu x.\alpha \qquad (x \in Var, a \in Act, 0 < r < 1)$$

A *transition probability* $\rho : Pr \times Act \times Pr \to [0,1]$ is the least fixed point of the recursive equation $\rho = P(\rho)$, where P is defined as follows:

$$P(\rho)(aP, b, Q) = \begin{cases} 1 & \text{if } b = a \text{ and } P \simeq Q \\ 0 & \text{otherwise} \end{cases}$$

$$P(\rho)(P_{1r} + P_2, a, Q) = r \cdot \rho(P_1, a, Q) + \bar{r} \cdot \rho(P_2, a, Q)$$

$$P(\rho)(\mu x.\alpha, a, Q) = \rho(\alpha[\mu x.\alpha/x], a, Q)$$

A *probabilistic bisimulation* is an equivalence relation $\simeq$ on processes that satisfies the following conditions:

- Whenever $P \simeq P'$, then for all actions a and all $E \in S/\simeq$, we have $\sum_{Q\in E}\rho(P,a,Q) = \sum_{Q'\in E}\rho(P',a,Q')$.

$\langle \mathcal{P}, w\rangle$ is bisimular to $\langle \mathcal{Q}, u\rangle$, written $\langle \mathcal{P}, w\rangle \simeq \langle \mathcal{Q}, u\rangle$, $\Leftrightarrow$ Suppose $\langle \mathcal{P}, w\rangle \simeq \langle \mathcal{Q}, u\rangle$:

- $\forall \alpha \in Act.\langle \alpha, w'\rangle \in \pi_2(e(w)).\exists \langle a, u'\rangle \in \pi_2(f(u))\& \sum_{w':\langle \mathcal{P}, w'\rangle \simeq \langle \mathcal{Q}, u'\rangle} \rho(\alpha, w, w') = \sum_{u':\langle \mathcal{P}, w'\rangle \simeq \langle \mathcal{Q}, u'\rangle} \sigma(\alpha, u, u')$.
- $\forall \varphi \in L(\Phi, \Pi).\rho(\varphi?, w, w) = \sigma(\varphi?, u, u)$

$\langle \mathcal{P}, w\rangle$ is bisimular to $\langle \mathcal{Q}, u\rangle$ except φ, written $\langle \mathcal{P}, w\rangle \simeq_\varphi \langle \mathcal{Q}, u\rangle$, $\Leftrightarrow$ Suppose $\langle \mathcal{P}, w\rangle \simeq_q \langle \mathcal{Q}, u\rangle$:

- $\forall \alpha \in Act.\langle \alpha, w'\rangle \in \pi_2(e(w)).\exists \langle a, u'\rangle \in \pi_2(f(u))\& \sum_{w':\langle \mathcal{P}, w'\rangle \simeq_\varphi \langle \mathcal{Q}, u'\rangle} \rho(\alpha, w, w') = \sum_{u':\langle \mathcal{P}, w'\rangle \simeq_q \langle \mathcal{Q}, u'\rangle} \sigma(\alpha, u, u')$.
- $\sigma(\varphi?, u, u) = 1$
- $\forall \psi(\neq \varphi) \in L(\Phi, \Pi).\rho(\psi?, w, w) = \sigma(\psi?, u, u)$

$\simeq$ is used to define the semantics of update action $\varphi!$.

Dynamic Probabilistic Logic

The language of DPrL is defined as follows:

Definition 3 *Let $p \in \Phi$, $r \in [0, 1]$. $\varphi \in L(\Phi, \Pi)$ (formulas) is defined as follows:*

$$\varphi ::= p|\varphi_1 \vee \varphi_2|\neg\varphi|[\alpha]\varphi|(P \geq r)\rangle\varphi$$

$\alpha \in Act(\Phi, \Pi)$ (actions) is defined as follows:

$$\alpha ::= \varphi!|\varphi?|\alpha_{1r} + \alpha_2|\alpha_1; \alpha_2|\alpha^*$$

The semantics of DPrL is defined based on a pair of a PKM $\mathcal{P} = \langle W, e, \{\rho(\alpha)\}_{\alpha \in Act(\Phi,\Pi)}\rangle$ and a state $w \in W$ as follows:

- $\mathcal{P}, w \models p \Leftrightarrow p \in e(w)\&\rho(p?)(w, w) = 1$
- $\mathcal{P}, w \models \varphi_1 \vee \varphi_2 \Leftrightarrow \mathcal{P}, w \models \varphi_1 \vee \mathcal{P}, w \models \varphi_2 \rho(\varphi_1 \vee \varphi_2?) = \rho(\varphi_1?) + \rho(\varphi_2?)$
- $\mathcal{P}, w \models \neg\varphi \Leftrightarrow \mathcal{P}, w \not\models \varphi\&\forall w, u \in W.\rho(\neg\varphi?)(w, u) = 1 - \rho(\varphi?)(w, u)$
- $\mathcal{P}, w \models (P \geq r)\varphi \Leftrightarrow \rho(\varphi?)(w, w) \geq r$
- $\mathcal{P}, w \models [\alpha]\varphi \Leftrightarrow \forall \langle \mathcal{P}', w'\rangle.\langle \mathcal{P}, w\rangle [\![\alpha]\!] \langle \mathcal{P}', w'\rangle \Rightarrow \mathcal{P}', w' \models \varphi$
- $\langle \mathcal{P}, w\rangle [\![\varphi?]\!] \langle \mathcal{P}, w\rangle \Leftrightarrow \mathcal{P}, w \models \varphi\&\rho(\varphi?)(w, w) = 1$
- $\langle \mathcal{P}, w\rangle [\![\alpha_{1r} + \alpha_2]\!] \langle \mathcal{P}', w'\rangle \Leftrightarrow \langle \mathcal{P}, w\rangle [\![\alpha_1]\!] \langle \mathcal{P}', w'\rangle\&\langle \mathcal{P}, w\rangle [\![\alpha_2]\!] \langle \mathcal{P}', w'\rangle\&\rho'(\alpha_1)(w, w') = r \cdot \rho(\alpha_1)(w, w')\&\rho'(\alpha_2)(w, w') = (1-r) \cdot \rho(\alpha_2)(w, w')$.
- $\langle \mathcal{P}, w\rangle [\![\alpha_1; \alpha_2]\!] \langle \mathcal{P}', w'\rangle \Leftrightarrow \exists \mathcal{P}'' \in PKS(\Gamma).w'' \in W_{\mathcal{P}''}\&\langle \mathcal{P}, w\rangle [\![\alpha_1]\!] \langle \mathcal{P}'', w''\rangle \&\langle \mathcal{P}'', w''\rangle [\![\alpha_2]\!] \langle \mathcal{P}', w'\rangle\&\rho'(\alpha_1; \alpha_2) = \rho''(\alpha_2) \circ \rho(\alpha_1)$.
- $\langle \mathcal{P}, w\rangle [\![\alpha^*]\!] \langle \mathcal{P}', w'\rangle \Leftrightarrow \langle \mathcal{P}, w\rangle \bigcup_{k=0}^{\infty}([\![\alpha]\!])^k \langle \mathcal{P}', w'\rangle\&\rho(\alpha^*) = \sum_{k=0}^{\infty}([\![\alpha]\!])^k$.
- $\langle \mathcal{P}, w\rangle [\![\varphi!]\!] \langle \mathcal{P}', w'\rangle \Leftrightarrow \langle \mathcal{P}, w\rangle \simeq_\varphi \langle \mathcal{P}', w'\rangle\&\rho' = \rho_\varphi$.
- *ρ_φ is defined as follows:*
  1. $\rho_p(\alpha)(w, u) = \begin{cases} 1 & \textit{if } u = w\&\alpha = p? \\ \rho(\alpha)(w, u) & \textit{otherwise} \end{cases}$
  2. $\rho_{\varphi_1 \wedge \varphi_2}(\alpha)(w, u) = \begin{cases} 1 & \textit{if } u = w\&\alpha = \varphi_1 \wedge \varphi_2, \varphi_1, \varphi_2 \\ \rho(\alpha)(w, u) & \textit{otherwise} \end{cases}$
  3. $\rho_{\neg\varphi}(\alpha)(w, u) = \begin{cases} 1 & \textit{if } u = w\&\alpha = \neg\varphi \\ \rho(\alpha)(w, u) & \textit{otherwise} \end{cases}$
  4. $\rho_{[\beta]\varphi}(\alpha)(w, u) = \begin{cases} 1 & \textit{if } u = w\&\alpha = [\beta]\varphi \\ \rho(\alpha)(w, u) & \textit{otherwise} \end{cases}$
  5. $\rho_{(P \geq r)\varphi}(\alpha)(w, u) = \begin{cases} 1 & \textit{if } u = w\&\alpha = (P \geq r)\varphi \\ \rho(\alpha)(w, u) & \textit{otherwise} \end{cases}$

Dynamic Conjecture Logic

I propose a dynamic conjecture logic by extending DPrL, called Dynamic Conjecture Logic (DCL), by adding probabilistic branchin time, in order to treat future conjectures or predictions related to time transition probabilities.

As we have seen in the previous section, the logic of branching time is CTL, which has formulas of forms of $\forall(\varphi_1 \mathcal{U} \varphi_2)$ and $[\forall X]\varphi$. However, since prediction is a conjecture about an future event, we need only formulas of form of $[\exists F]\varphi$, which means for some time path, for some future time on the path, φ holds at the time. Therefore, DCL is DPrL + $[\exists F]\varphi$ + a branching time structure + epistemic modality + an epistemic structure.

Given a probabilistic Kripke system $\langle W, e, \rho\rangle$, I extend ρ to $\rho \cup \rho(<)$, where $\rho(<) : W \times W \to [0, 1]$ and $e : W \to pow(\Phi) \times pow(Act \times W)$ to $e : W \to pow(\Phi) \times pow((Agent \cup Act) \times W)$. $\Omega_w = \{w_0 w_1 \ldots w_n | w_0 = w, \rho(<)(w_i, w_{i+1}) > 0, i \in \omega\}$.

The semantics of $[\exists F]\varphi$ is defined as follows:

$$\mathcal{P}, w \models [\exists F]\varphi \Leftrightarrow \exists \sigma \in \Omega_w.\exists i \in \omega.\mathcal{P}, \sigma_i \models \varphi$$

Furthermore, *DCL* is added the following formula:

$$S_a\varphi$$

which means 'agent a feels that the happening of φ is significant or remarkable'.

Its semantics is defined as follows:

$$\begin{aligned} & \mathcal{P}, w \models S_a\varphi \\ \Leftrightarrow\ & \mathcal{P}, w \models (P \geq r)\varphi \\ & \&\forall \langle a, u\rangle \in e(w).\frac{r}{\epsilon_a(\varphi, w)} > 1, \end{aligned}$$

where $\epsilon_a(\varphi) = s$ iff $\mathcal{P}, w \models B_a(P \geq s)\varphi$, i.e., ϵ_a denotes agent a's expectation of happening of φ. S_a is defined by the probability normalized by a's expectation.

For example, $[\varphi?]S_a\psi$ means that holding φ is a chance of ψ for a, and $[\varphi!]S_a\psi$ means that getting data φ is a chance of ψ for a.

Discovery of Subjective Chances

By the term *subjective chances*, I mean the 'chance' in the following contexts (1), but not (2):

(1) He missed a chance to win \$ 25 from them.

(2) a. There is a 30% chance of an earthquake tomorrow.

b. His chances of promotion are very good.

In other words, the 'subjective chances' means the situations or events, with their types (propositions), which causes something. On the other hand, 'chance' in the context (2), I will call it the *objective chances*, i.e., the probability or the degrees of expectation.

As tangible examples, I will show some cases characterizing subjective chances below: a puzzle on book buying, a puzzle on refuge, and Mitsunari's lesson.

A Puzzle on Book Buying

(3) When a man was irresolute whether to buy a book which is needed for his work or not, the salesclerk said, "This copy is the last one". The probability of his buying the book is stable, i.e., 0.5, before and after the clerk's saying, but we can guess that he would buy it after he heard so. Why?

This example shows the difference between subjective chances and objective chances. Whereas subjective chances are affected by agents' epistemic changes, objective chances are not.

In this case, the clerk's utterance can be a chance to buy the copy of the book. In DCL, this situation can be formalized as follows:

$$([(\varphi!; [\exists F]buy?)_r + \neg[\exists F]buy?]\top) \wedge [\varphi!]S_a buy$$

Mitsunari's Lesson A Japanese samurai general, Mitsunari Ishida, lost one of the largest battle in Japan, though his army was overwhelming against his enemy. Some historians point out that he had at least three chances to win the battle during fighting, although he lost all of them. This episode can be formalized as follows:

- $s \models \varphi_1 \rightarrow [\exists F]win$
- $s \models \neg[\varphi_1!]S_a win \wedge [\exists F](\varphi_2 \rightarrow [\exists F]win)$
- $s \models \neg[\varphi_1!]S_a win \wedge [\exists F](\neg[\varphi_2!]S_a win \wedge [\exists F](\varphi_3 \rightarrow [\exists F]win))$

Then "his chances" are:

- $s \models \varphi_1$
- $s < s', s' \models \varphi_2$
- $s' < s'', s'' \models \varphi_3$

These chances are described in DCL as follows:

$$\begin{array}{l}([\varphi_1! \ \mathsf{or}_r \ [\exists F]([\varphi_2! \ \mathsf{or}_{r'} \ [\exists F][\varphi_3!]\top?]\top?)])[\exists F]win \\ \wedge\neg[\varphi!]S_a win \\ \wedge\neg[\neg[\varphi!]S_a win?; \varphi!]S_a win \\ \wedge\neg[\neg[\neg[\varphi!]S_a win?; \varphi!]S_a win?; \varphi!]S_a win\end{array}$$

where $[\varphi \ \mathsf{or}_r \ \alpha] \equiv [\varphi!_r + (\neg\varphi?; \alpha)]$. Therefore, the discovery of the chances of φ for agent a is formalized as in the following deduction schema:

$$\text{find } \alpha \ \text{ such that } \Gamma \vdash [\alpha]S_a[\exists F]\varphi$$

where Γ is a given context.

Conclusion

As we have seen, in this paper, I have survey probabilistic logics and try to combine probabilistic logic and dynamic epistemic logics by considering some probabilistic transition systems. Many types of probabilistic logics are summarized, characterized, and we could find some common properties of them, which can be exploited to propose a dynamic probabilistic logic (DPrL). As a result, the proposed logic DPrL can treat probabilistic belief revision and its extension DCL can formalize discovery of subjective chances by considering some cases of discovery of subjective chances.

Although these logics have only their syntax and semantics based on probabilistic Kripke systems, if their model checking algorithms are defined, these systems can be applied to prediction, discovery of chances and their verification. This task is my future work.

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